

THE CALCULATION OF PENETRATION INDICES FOR EXCEPTIONAL WILD ARCS

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ABSTRACT. A new class of wild arcs is defined, the class of "exceptional" arcs, which is a subclass of the class of arcs whose only wildpoint is an endpoint. This paper then uses geometric techniques to calculate the penetration indices of these exceptional arcs.

Finding the penetration index of an arc or a simple closed curve at its only wildpoint is, in the general case, a difficult task (cf. [2], [8]). For arcs, we have a set of conditions which ensure that the arc will be wild at one endpoint [5], but no way of computing the penetration index of the arc at that point. The object of this paper is to define a class of arcs which are wild at one endpoint ("nearly polyhedral" [2]), which satisfy the hypotheses of the theorem of [5], and whose penetration index may be easily calculated.

The results of this paper are part of the author's doctoral thesis [4], and the techniques used here will be of fundamental importance in finding invariants of oriented local type for exceptional arcs (see [6, paper 3]).

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1. The notations Bd , Cl , Int , $N(X)$, $\nu(k \cap X)$ are all defined in [5] and [6] and, generally speaking, the notation and terminology of this paper will be consistent with that of those papers. In particular, we have the following conventions and definitions:

1. k is an oriented arc in Euclidean 3-space R^3 , and is tame at all points except its endpoint p . In this paper, the precise orientation of k does not concern us, for the orientation has only been introduced so that we can use arguments based on the algebraic intersection number of k with a surface.

2. A handlebody of genus g is a tame closed regular neighbourhood of a wedge of g circles in R^3 . If $g = 1$, such a handlebody is a solid torus (i.e. the product of S^1 with a 2-disc—called "Vollring" in [7]), and if $g = 0$ we have a 3-cell.

The penetration index $P_g(k, p)$ of k at p , relative to handlebodies of genus g , is the smallest integer n such that there exist arbitrarily small neighbourhoods of p which are handlebodies of genus g , each meeting k on its boundary in n points. When there is no danger of confusion, we shall write P_g instead of $P_g(k, p)$. P_1 is

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the toral penetration index, and P_0 is the “nice penetration index” of [3].

Note that if $g(k)$ is the smallest integer r such that $P_r(k, p) = 1$, then $g(k) \geq \text{LEG}(k, p)$ [8, Definition (2.3)], with equality if k is wild and $g(k) = 1$. It is not known whether $g(k) = \text{LEG}(k, p)$ for all k .

Henceforth, we shall assume that $P_1(k, p) = 1$.

3. We choose a 3-cell neighbourhood E_0 of p , which meets k on its boundary in at least three points, and so that if $E \subset \text{Int } E_0$ is any other 3-cell neighbourhood of p , the surface $\text{Bd } E$ meets k in at least as many points as does $\text{Bd } E_0$. If $P_0(k, p)$ is finite, we require that $\text{Bd } E_0$ meet k in P_0 points. All our working takes place inside E_0 .

A k -torus is a tame closed solid torus $V \subset \text{Int } E_0$ which contains p in its interior, and meets k on its boundary in one point only. If U and V are k -tori and $U \subset \text{Int } V$, we write $U < V$ iff U lies in some 3-cell in the interior of V .

A constructing sequence for k (in E_0) is a sequence of k -tori $E_0 \supset V_0 > V_1 > V_2 > \dots$ such that

- (i) $\cap V_i = p$;
- (ii) if U is any k -torus with $V_h \subset \text{Int } U \subset U \subset \text{Int } V_{h-1}$ for some $h (\geq 1)$, then either V_h has nonzero order in U (cf. [6] or [7, p. 172 ff.]) or U has nonzero order in V_{h-1} ; if U is any k -torus which contains V_0 in its interior, we require V_0 to have nonzero order in U .

4. k is *exceptional* if there exists a sequence $\mathcal{C} : E_0 \supset V_0 \supset E_1 \supset V_1 \supset E_2 \supset V_2 \supset \dots$ of k -tori and tame closed 3-cells such that

- (i) $E_0 \supset V_0 > V_1 > V_2 > \dots$ is a constructing sequence for k in E_0 ;
- (ii) for each $i \geq 1$, $\text{Bd } E_i \subset \text{Int}(V_{i-1} - V_i)$;
- (iii) the sequence $E_0 \supset E_1 \supset E_2 \supset E_3 \supset \dots$ satisfies the hypotheses of the theorem of [5].

Condition (iii) means that the sets $A(E_i, E_{i+1})$ (of those subarcs of k in $E_i - E_{i+1}$ whose endpoints both lie on $\text{Bd } E_i$) and $B(E_{i+1}, E_i)$ (of those subarcs of k in $E_i - \text{Int } E_{i+1}$ whose endpoints both lie on $\text{Bd } E_{i+1}$) are not empty for any i ; that given any $\alpha \in A(E_i, E_{i+1})$ there exists a $\beta \in B(E_{i+1}, E_i)$ such that the pair (α, β) is unsplittable (in the sense of [5]), and given any $\beta \in B(E_{i+1}, E_i)$ there exists an $\alpha \in A(E_i, E_{i+1})$ such that the pair (α, β) is unsplittable. Thus condition (iii) guarantees that exceptional arcs are nearly polyhedral. Note also that $\text{LEG}(k, p) = 1$.

The sequence \mathcal{C} is called a *special constructing sequence* for k in E_0 .

(In [4], the exceptional arcs are called “special”.)

2. For each pair of indices h and $H > h$, we define the local penetration index of k in the region $V_h - V_H$, $P_0(k, V_h - V_H)$, to be the smallest integer n such that there exists a tame closed 3-cell neighbourhood E of p whose boundary lies in $\text{Int}(V_h - V_H)$ and meets k in n points. Note that:

- (a) if P_0 is finite, $P_0(k, p) \leq P_0(k, V_h - V_H)$ because every 3-cell neighbourhood of p in $\text{Int } E_0$ meets k on its boundary in at least $P_0(k, p)$ points, by assumption

(see number 3);

(b) if $l \leq h$ and $L \geq H$, $P_0(k, V_l - V_L) \leq P_0(k, V_h - V_H)$; and

(c) $P_0(k, V_h - V_H) \leq \min\{P_0(k, V_j - V_{j+1}) : h \leq j \leq H - 1\}$.

Our goal in this section is the proof of the following theorem, which states that we have strict equality in (c).

Theorem 1. *Let k be a nearly polyhedral arc which has a constructing sequence $E_0 \supset V_0 > V_1 > V_2 > \dots$. Then for each pair of integers h, H with $h < H$, there exists an index s , $h \leq s < H$, such that $P_0(k, V_h - V_H) = P_0(k, V_s - V_{s+1})$.*

Before proving this theorem, we need the following lemmas:

Lemma 1. *If V is a k -torus, every meridian disc of V meets k in at least one point (we assume that our meridian discs do not contain p). If V is unknotted, every disc bounded by a longitude of $\text{Bd } V$ meets k in at least one point. (The proof is similar to the proof of Lemma 1 in paper 2 of [6].)*

The proof of the following lemma is easy, and is therefore omitted.

Lemma 2. *Let R and R' be annuli with boundary curves σ, τ and σ', τ' respectively. Let S be a surface obtained by identifying σ with σ' and τ with τ' . Then σ does not bound any surface on S .*

Lemma 3. *Let E_0 be any 3-cell, and E a tame 3-cell in $\text{Int } E_0$. Let $R \subset \text{Int } E_0 - \text{Int } E$ be a tame annulus which intersects $\text{Bd } E$ only in the two curves σ and τ of $\text{Bd } R$. σ and τ together bound an annulus $R' \subset \text{Bd } E$; let D_σ and D_τ be the disjoint discs of $\text{Cl}(\text{Bd } E - R')$ which are bounded by σ and τ respectively.*

Then if E' is the 3-cell bounded by $R \cup D_\sigma \cup D_\tau$, E' contains E iff $R \cup R'$ is not the boundary of a solid torus V which contains E and in which D_σ and D_τ are meridian discs.

Proof. We shall prove that E is not contained in E' iff $R \cup R'$ is the boundary of such a solid torus V .

(If.) If $R \cup R'$ bounds such a solid torus, the closures of the components of $V - D_\sigma \cup D_\tau$ are the 3-cells E and E' . That is, $V = E \cup E'$, $E \cap E' = D_\sigma \cup D_\tau$, and $E \not\subset E'$.

(Only if.) We wish to prove that if E is not contained in E' , then $R \cup R'$ is the boundary of a solid torus of the kind specified by the lemma.

Let $R \cup R' = \text{Bd } V$, where $V \subset \text{Int } E_0$ is either a solid torus or a 3-cell with a knotted hole. We collapse $\text{Bd } E_0$ to a point and obtain a 3-sphere $S_0 = E_0 / \text{Bd } E_0$; if V is not a solid torus, the image of $\text{Cl}(E_0 - V)$ under the collapsing map is a solid torus V' which contains $S_0 - E$, and $V' \cup V = S_0$ (see [1]).

By Lemma 2, σ is not null-homologous on $\text{Bd } V$, yet bounds a disc D_σ in the closure of one component of $S_0 - \text{Bd } V$. Therefore σ is either a meridian or a

longitude of the closure of whichever component of $S_0 - \text{Bd } V$ is an open solid torus. We have four possibilities to consider, depending whether σ is (i) a longitude of V , (ii) a meridian of V' , (iii) a longitude of V' , or (iv) a meridian of V . Since we are assuming that $E \not\subset E'$, we will show that (iii) and (iv) are the only possible choices for σ , then show that this implies that $R \cup R'$ is the boundary of a solid torus (namely V) which contains E , and in which D_o and D_r are meridian discs.

(i) σ is a longitude of V . Then D_o (and hence E) cannot lie in V , and V must be unknotted. By [1], σ is then a meridian of V' ; that is, case (ii) applies.

(ii) σ is a meridian of V' . D_o and D_r separate V' into two 3-cells bounded by $R \cup D_o \cup D_r$ and $R' \cup D_o \cup D_r$, and one of these 3-cells contains the complement of E .

If $V' = E \cup (S_0 - E')$, then $E \subset E'$; whereas if $V' = E' \cup (S_0 - E)$, E' lies in E and so $R \subset E$, which contradicts the hypothesis on R . So $V' = E \cup (S_0 - E')$ and $E \subset E'$.

Since we are assuming that $E \not\subset E'$, it follows that σ must either be a longitude of V' or a meridian of V .

(iii) σ is a longitude of V' . Then V' must be unknotted; by [1], $V = \text{Cl}(S_0 - V')$ is an unknotted solid torus which has σ as a meridian, so case (iv) applies. Note that D_o is a meridian disc of V and that $\text{Bd } V = R \cup R'$.

(iv) σ is a meridian of V . D_o is a meridian disc of the solid torus V and similar arguments show that τ is also a meridian of V , bounding the meridian disc D_r . Thus $E \subset V$.

Thus, if $E \not\subset E'$, $R \cup R'$ must be the boundary of a solid torus which contains E , and in which D_o and D_r are meridian discs. Q.E.D.

Now we may proceed to the proof of Theorem 1.

Let \mathcal{B} be the class of all tame closed 3-cell neighbourhoods of p whose boundaries lie in $\text{Int}(V_k - V_H)$ and meet k in $P_0(k, V_k - V_H)$ points. There exists at least one 3-cell $E \in \mathcal{B}$ such that

(i) the surface $\text{Bd } E$ is in general position with respect to the surfaces $\text{Bd } V_{h+1}, \dots, \text{Bd } V_{H-1}$, and none of the "intersection curves" of $\text{Bd } E \cap \bigcup_{j=h+1}^{H-1} \text{Bd } V_j$ contains any points of k ;

(ii) the number of intersection curves is minimal in \mathcal{B} with respect to (i).

To prove the theorem, it is sufficient to prove that the number of intersection curves is zero. Assume to the contrary, and let s be the smallest index such that $\text{Bd } E \cap \text{Bd } V_s \neq \emptyset$.

The proof of the theorem is in three parts.

Part 1. Suppose some intersection curve bounds a disc on $\text{Bd } V_s$. We may choose one such curve, σ say, which bounds a disc $D \subset \text{Bd } V_s$ containing no other intersection curves. σ also bounds a disc $D' \subset \text{Bd } E$ which together with D bounds a 3-cell C which does not contain p .

Let N be a closed regular neighbourhood of C in E_0 ; put $E' = \text{Cl}(E - N)$ if $D \subset E$, otherwise put $E' = E \cup N$. Then E' is a tame closed 3-cell neighbour-

hood of p . By taking a small enough N , we may ensure that $\text{Bd } E'$ lies in $\text{Int}(V_h - V_H)$, is in general position with respect to the surfaces $\text{Bd } V_{h+1}, \dots, \text{Bd } V_{H-1}$, and meets this family of surfaces in fewer intersection curves than did $\text{Bd } E$. For we can ensure that

$$\text{Bd } E' \cap \text{Bd } V_j \subset \text{Bd } E \cap \text{Bd } V_j$$

unless $j = s$, when

$$\text{Bd } E' \cap \text{Bd } V_s \subset \text{Bd } E \cap \text{Bd } V_s - \{\sigma\},$$

thus eliminating all those intersection curves that lie in D' . Then, by choosing a smaller regular neighbourhood if necessary, we may guarantee that $N(k \cap \text{Bd } N) = N(k \cap \text{Bd } C)$ by requiring that the set $k \cap \text{Cl}(N - C)$ consist of precisely $N(k \cap \text{Bd } C)$ arcs running from $\text{Bd } C$ to $\text{Bd } N$, or from $\text{Bd } N$ to $\text{Bd } C$. We will call N a "suitably small regular neighbourhood of C ", and in future the phrase "suitably small (or sufficiently small) regular neighbourhood of the set X " will designate a regular neighbourhood of X which has this nicety of position with respect to k , X , and any other surfaces that may be nearby.

By choosing a sufficiently small regular neighbourhood N of C , we can obtain a 3-cell E' which satisfies the condition (i) satisfied by E , as described in the preceding paragraph. We will obtain a contradiction to our choice of E , using condition (ii), if we can show that $E' \in \mathcal{B}$. It only remains to compare $N(k \cap \text{Bd } E')$ with $N(k \cap \text{Bd } E)$ and show that $\text{Bd } E'$ meets k in $P_0(k, V_h - V_H)$ points. Note that

$$N(k \cap \text{Bd } E') = N(k \cap \text{Bd } E) - N(k \cap D') + N(k \cap D),$$

and since $N(k \cap \text{Bd } E)$ is minimal, it follows that

$$N(k \cap D') \leq N(k \cap D) \quad (\leq P_1 = 1).$$

Since D lies on the boundary of V_s , k meets D in at most one point; k cannot meet D' at all if $k \cap D = \emptyset$, and

$$N(k \cap \text{Bd } E') = N(k \cap \text{Bd } E) = P_0(k, V_h - V_H)$$

in this case. So suppose k meets D in one point.

Because $p \notin C$, $\nu(k \cap \text{Bd } C) = 0$, where $\text{Bd } C$ inherits its orientation from R^3 . Thus

$$0 = \nu(k \cap \text{Bd } C) = \nu(k \cap D) + \nu(k \cap D') = \pm 1 + \nu(k \cap D')$$

which implies that k meets D' in an odd number of points. Hence k must meet D' in precisely one point, so again

$$N(k \cap D) = N(k \cap D') \quad \text{and} \quad N(k \cap \text{Bd } E') = P_0(k, V_h - V_H).$$

The preceding paragraphs show that the 3-cell E' lies in the class \mathcal{B} which, as remarked above, contradicts the minimality assumption (ii) involved in our choice of E . It follows that no curve of $\text{Bd } E \cap \text{Bd } V_j$ can be null-homologous on $\text{Bd } V_j$.

Since the intersection curves of $\text{Bd } E \cap \text{Bd } V_j$ all bound discs on $\text{Bd } E$, these curves are either all meridians of $\text{Bd } V_j$, or all longitudes. In either case, we have an even number of intersection curves bounding parallel annuli on $\text{Bd } V_j$.

We may therefore choose a pair of curves σ and τ which bound an annulus $R \subset \text{Bd } V_j$ whose interior lies in the interior of E , and therefore contains no intersection curves. R separates E into two components: a 3-cell C and a "complementary space" $K(R) = K$ which is either a solid torus or a 3-cell with a knotted hole. If $p \in C$, and N is a suitably small open regular neighbourhood of K in E , the 3-cell $E' = E - N$ is a tame closed neighbourhood of p whose boundary lies in $\text{Int}(V_h - V_H)$ and therefore meets k in at least as many points as $\text{Bd } E$ does; that is

$$N(k \cap \text{Bd } E') \geq N(k \cap \text{Bd } E) = P_0(k, V_h - V_H).$$

Let R' be the closure of the annular component of $\text{Bd } E - \{\sigma, \tau\}$. Then we have the inequality

$$N(k \cap R') \leq N(k \cap R) \leq P_1 = 1;$$

as before, this implies that $N(k \cap R') = N(k \cap R)$ if k does not meet R at all, and in this case $\text{Bd } E'$ meets k in precisely $P_0(k, V_h - V_H)$ points. So suppose k meets R in one point.

Because $p \notin K$, $\nu(k \cap \text{Bd } K) = 0$, so

$$0 = \nu(k \cap \text{Bd } K) = \nu(k \cap R) + \nu(k \cap R') = \pm 1 + \nu(k \cap R').$$

This implies that k meets R' in an odd number of points which, from the above, can be no larger than one; so again

$$N(k \cap R) = N(k \cap R') \quad \text{and} \quad N(k \cap \text{Bd } E') = P_0(k, V_h - V_H).$$

Further, $\text{Bd } E'$ is in general position with respect to the surfaces $\text{Bd } V_{h+1}, \dots, \text{Bd } V_{H-1}$, and meets these surfaces in fewer intersection curves than does $\text{Bd } E$. For

$$\text{Bd } E' \cap \text{Bd } V_j \subset \text{Bd } E \cap \text{Bd } V_j$$

unless $j = s$, when

$$\text{Bd } E' \cap \text{Bd } V_s \subset \text{Bd } E \cap \text{Bd } V_s - \{\sigma, \tau\}$$

—in particular, those intersection curves which lie in R' have been eliminated. The 3-cell E' lies in the class \mathcal{B} , and its existence contradicts the minimality assumption involved in the choice of our original 3-cell E . We must assume, therefore, that for each such annulus R , the point p lies in $K(R)$ and not in C . Then the statement P below is true.

P. p lies in the complementary space of each annulus $R \subset E \cap \text{Bd } V_s$ whose interior lies in the interior of E .

Part 2. Here we prove that, if necessary, we can “thicken” the 3-cell E by replacing an annulus $R' \subset \text{Bd } E$ by a suitable annulus R which lies in $\text{Bd } V_j \cap \text{Cl}(E_0 - E)$ (Lemma 3 is important here). That we can find such a suitable annulus R , so that the 3-cell so obtained contains p in its interior, is the gist of Sublemma 2.

Sublemma 1. *If $\text{Bd } E \cap \text{Bd } V_s \neq \emptyset$, all the intersection curves are nested about one disc on $\text{Bd } E$. That is, there are only two intersection curves which bound discs on $\text{Bd } E$ which contain no other intersection curves.*

Proof. Suppose the intersection curves are all meridians of $\text{Bd } V_s$. Let D be a meridian disc of V_s such that $N(k \cap D) = n \geq 1 = P_1$ is minimal (cf. Lemma 1). Then $V_s - \{\text{a suitably small open regular neighbourhood of } D\}$ is a 3-cell neighbourhood of p which meets k on its boundary in $2n + P_1$ points, so

$$2n + P_1 \geq P_0(k, V_s - V_H) = N(k \cap \text{Bd } E).$$

Suppose there are $2r$ disjoint meridian discs D_1, \dots, D_{2r} of V_s lying on $\text{Bd } E$. Then

$$2n + P_1 \geq N(k \cap \text{Bd } E) \geq \sum_{j=1}^{2r} N(k \cap D_j) \geq 2rn;$$

that is, $(2r - 2)n \leq P_1 = 1$. Since $n \geq 1$, $r = 1$ and only two intersection curves can bound discs on $\text{Bd } E$ which are meridian discs of V_s .

A similar argument shows that only two intersection curves can bound discs on $\text{Bd } E$ which are longitude discs of V_s . Q.E.D.

Let the intersection curves be $\sigma_1, \dots, \sigma_n$, and let us assume that σ_1 bounds a disc D_1 on $\text{Bd } E$ which contains no other intersection curves. Each curve σ_i , for $i \geq 2$, bounds two discs on $\text{Bd } E$ and one of these, D_i say, contains D_1 in its interior. By Sublemma 1, it is possible to number the intersection curves so that $D_1 \subset D_2 \subset \dots \subset D_n$ (with $\sigma_i = \text{Bd } D_i$). [Note that this implies that the disc $\text{Cl}(\text{Bd } E - D_n)$ contains no intersection curves in its interior.]

Before stating Sublemma 2, it is necessary to introduce some notation. For each i , we denote $\text{Cl}(\text{Bd } E - D_i)$ by D'_i (so $D'_n \subset D'_{n-1} \subset \dots \subset D'_1$). For each pair i and j with $i < j$, we use R'_{ij} to denote the annulus $\text{Cl}(\text{Bd } E - D_i \cup D'_j)$ bounded by σ_i and σ_j on $\text{Bd } E$. If σ_i and σ_j also bound an annulus on $\text{Bd } V_s$ whose interior lies in $\text{Int}(V_s - E)$ (and therefore contains no intersection curves), we shall denote this annulus by R_{ij} .

Sublemma 2. *If there are four or more intersection curves of $\text{Bd } E \cap \text{Bd } V_s$, there exists a pair of curves σ_i and σ_j such that the 3-cell bounded by $R_{ij} \cup D_i \cup D'_j$ contains p in its interior.*

Proof. Let n be the number of curves of $\text{Bd } E \cap \text{Bd } V_s$; we are assuming $n \geq 4$.

According to Lemma 3, we need only show that for some pair $i < j$, $R_{ij} \cup R'_{ij}$ is not the boundary of a solid torus which contains E and in which D_i and D'_j are meridian discs.

Suppose $R_{ij} \cup R'_{ij}$ is the boundary of a solid torus T_i which contains E and in which D_i and D'_j are meridian discs.

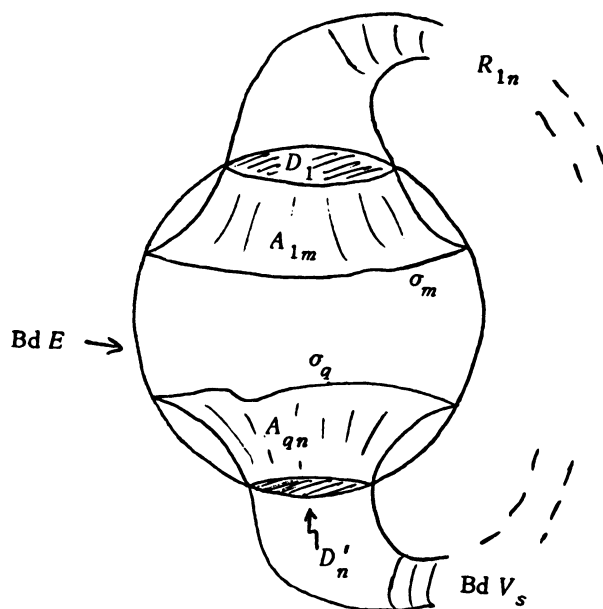


Figure 1

If $r = n$, E and $\text{Bd } V_s$ must intersect as shown in Figure 1, for there are no intersection curves in either $\text{Int } D_1$ or $\text{Int } D'_n$. Let A_{1m} be the annulus on $\text{Bd } V_s$ which has σ_1 as one of its boundary curves, and which lies inside E as shown; let σ_m be the other boundary curve of A_{1m} . Since $n \geq 4$, it follows that $m \neq n$. If C_m is the 3-cell component of $E - A_{1m}$, then $K(A_{qn}) = K_q$ lies in the interior of C_m ,

where K_q is the complementary space of the annulus A_{qn} shown in Figure 1. If we denote $K(A_{1m})$ by K_m , p must lie in $K_m \cap K_q$ because $\text{Bd } E$ and $\text{Bd } V_j$ intersect so that the statement P (see Part 1) is true. But $K_m \cap K_q = \emptyset$ because $K_q \subset \text{Int } C_m$.

This contradiction shows $r \neq n$. There is therefore an annulus $R_{ln} \subset \text{Bd } V_j$ whose interior lies in $\text{Int}(V_h - E)$ and whose boundary curves are σ_l and σ_n ; then $l > r$, so $\sigma_l \subset \text{Int } D'_r$.

Suppose that $R_{ln} \cup R'_{ln}$ is the boundary of a solid torus $T_n \supset E$ with meridian discs D_l and D'_n . Then $\text{Bd } T_l$ and $\text{Bd } T_n$ are disjoint, for R_l and R_{ln} are disjoint, and

$$R'_{ln} \subset D'_l \subset \text{Int}(\text{Bd } E - R'_{lr}).$$

Since D_l lies in the interior of D'_r , which is a meridian disc of T_n , it follows that $T_l \subset \text{Int } T_n$. On the other hand, D'_n is also a meridian disc of T_n , and D'_n lies in the interior of D'_r ; since D'_r is a meridian disc of T_l , $T_n \subset \text{Int } T_l$.

This situation is impossible; $R_{ln} \cup R'_{ln}$ cannot be the boundary of a solid torus containing E if $R_{lr} \cup R'_{lr}$ already bounds such a torus. Therefore at least one of the pairs σ_l , σ_r and σ_l , σ_n satisfies the conclusions of the sublemma. Q.E.D.

If $\text{Bd } E \cap \text{Bd } V_j \neq \emptyset$ and consists of more than two intersection curves, we may "thicken" E by replacing R'_{ij} by R_{ij} , for a suitable pair i and j . $R_{ij} \cup D_i \cup D'_j$ is the boundary of a 3-cell neighbourhood of p so, as before, $N(k \cap R_{ij}) = N(k \cap R'_{ij})$. Then if E' is a suitably small closed regular neighbourhood of the 3-cell bounded by $R_{ij} \cup D_i \cup D'_j$, E' is a tame closed 3-cell neighbourhood of p whose boundary lies in $\text{Int}(V_h - V_H)$, meets k in $N(k \cap \text{Bd } E) = P_0(k, V_h - V_H)$ points, and is in general position with respect to the family of surfaces $\text{Bd } V_{h+1}, \dots, \text{Bd } V_{H-1}$. The 3-cell E' therefore lies in the class \mathcal{B} . But $\text{Bd } E'$ meets the family of surfaces $\text{Bd } V_{h+1}, \dots, \text{Bd } V_{H-1}$ in fewer intersection curves than does $\text{Bd } E$, for

$$\text{Bd } E' \cap \text{Bd } V_j \subset \text{Bd } E \cap \text{Bd } V_j$$

unless $j = s$, when

$$\text{Bd } E' \cap \text{Bd } V_j \subset \text{Bd } E \cap \text{Bd } V_j - \{\sigma_i, \sigma_j\},$$

because all the intersection curves lying on R'_{ij} have been eliminated.

Part 3. The existence of the 3-cell $E' \in \mathcal{B}$ contradicts the minimality assumption involved in our choice of E if there are four or more intersection curves of $\text{Bd } E \cap \text{Bd } V_j$, so we conclude that $\text{Bd } E \cap \text{Bd } V_j$ consists of at most two curves σ and τ . σ and τ bound disjoint discs D_σ and D_τ on $\text{Bd } E$, and we denote by R' the annulus $\text{Cl}(\text{Bd } E - D_\sigma \cup D_\tau)$. We then have two cases to consider: (a) σ and τ are longitudes of $\text{Bd } V_j$; (b) σ and τ are meridians of $\text{Bd } V_j$.

(a) σ and τ are longitudes of $\text{Bd } V_j$. Let R be the annulus $\text{Bd } V_j \cap \text{Cl}(E_0 - E)$.

Then R' lies in V_j , and $R' \cap \text{Bd } V_j = \text{Bd } R' = \text{Bd } R$ so, by [7, Satz 1, p. 207],

$R' \cup (\text{Bd } V_s - R)$ is the boundary of a solid torus T which has order 1 in V_s and contains p in its interior (because $T = E \cap V_s$). Since k meets $\text{Bd } V_s$ in one point, it follows that $N(k \cap R') \geq N(k \cap R)$.

On the other hand, $E \cup V_s$ is a tame closed 3-cell neighbourhood of p whose boundary $(= R \cup D_o \cup D_r)$ lies in $\text{Int}(V_h - V_H)$, and therefore meets k in at least $P_0(k, V_h - V_H) = N(k \cap \text{Bd } E)$ points, so $N(k \cap R) \geq N(k \cap R')$.

Let E' be a suitably small closed regular neighbourhood of $E \cup V_s$; then E' has the properties

- (i) $N(k \cap \text{Bd } E') = N(k \cap \text{Bd } E) = P_0(k, V_h - V_H)$;
- (ii) $\text{Bd } E' \subset \text{Int}(V_{s-1} - V_s) \subset \text{Int}(V_h - V_H)$;
- (iii) $\text{Bd } E' \cap \bigcup_{j=h+1}^{H-1} \text{Bd } V_j = \emptyset$.

In particular, the 3-cell E' lies in the class \mathcal{B} , and property (iii) shows that the existence of E' contradicts our choice of the 3-cell E in \mathcal{B} . We must therefore conclude that σ and τ are not longitudes of $\text{Bd } V_s$.

(b) σ and τ are meridians of $\text{Bd } V_s$. Let R be the annulus $E \cap \text{Bd } V_s$. Since $R' \subset \text{Cl}(E_0 - V_s)$ and $R' \cap \text{Bd } V_s = \text{Bd } R' = \text{Bd } R$, $R' \cup (\text{Bd } V_s - R)$ is the boundary of a solid torus T which contains V_s and is therefore a neighbourhood of p (see [7, Satz 3, p. 215]). Because $N(k \cap \text{Bd } V_s) = 1$ is minimal, $N(k \cap R') \geq N(k \cap R)$.

On the other hand, $R \cup D_o \cup D_r$ is the boundary of a 3-cell neighbourhood of p , namely $E \cap V_s$, so $N(k \cap R) \geq N(k \cap R')$.

Let E' be the 3-cell $E \cap V_s - \{\text{a suitably small open regular neighbourhood of } R\}$. Then $\text{Bd } E'$ meets k in precisely $P_0(k, V_h - V_H)$ points, lies in $\text{Int}(V_s - V_H)$ and therefore in $\text{Int}(V_h - V_H)$, and is in general position with respect to the surfaces $\text{Bd } V_{s+1}, \dots, \text{Bd } V_{H-1}$. Therefore $E' \in \mathcal{B}$. However

$$\begin{aligned} \text{Bd } E' \cap \bigcup_{j=h+1}^{H-1} \text{Bd } V_j &= \text{Bd } E \cap \bigcup_{j=s+1}^{H-1} \text{Bd } V_j \\ &\subset \text{Bd } E \cap \bigcup_{j=s}^{H-1} \text{Bd } V_j - \{\sigma, \tau\} \end{aligned}$$

which contradicts the choice of our 3-cell E . We conclude that σ and τ cannot be meridians of $\text{Bd } V_s$.

It follows that $\text{Bd } E$ cannot meet $\text{Bd } V_s$ at all; but s was chosen to be the smallest index such that $\text{Bd } E \cap \text{Bd } V_s \neq \emptyset$. No such index s can exist, therefore; so

$$\text{Bd } E \cap \bigcup_{j=h+1}^{H-1} \text{Bd } V_j = \emptyset.$$

Since $\text{Bd } E \subset \text{Int}(V_h - V_H)$,

$$\text{Bd } E \cap \bigcup_{j=h}^H \text{Bd } V_j = \emptyset.$$

Therefore there exists an index s , $h \leq s < H$, such that $\text{Bd } E \subset \text{Int}(V_s - V_{s+1})$; since $N(k \cap \text{Bd } E) = P_0(k, V_h - V_H) \leq P_0(k, V_s - V_{s+1}) \leq N(k \cap \text{Bd } E)$, the theorem follows. Q.E.D.

Theorem 1 yields the following corollary.

Corollary. *Let k be a nearly polyhedral arc which has a constructing sequence $E_0 \supset V_0 \supset V_1 \supset V_2 \supset \dots$. Then k has finite nice penetration index at p iff there exists at least one integer m such that $P_0(k, V_s - V_{s+1}) = m$ for infinitely many indices s , in which case $P_0(k, p)$ is the smallest such m .*

3. We now apply these results to the study of exceptional arcs. The main theorem is the following:

Theorem 2. *Let k be an exceptional arc and $\mathcal{C} : E_0 \supset V_0 \supset E_1 \supset V_1 \supset E_2 \supset V_2 \supset \dots$ a special constructing sequence for k in E_0 . Define the set \mathcal{N} by*

$$\mathcal{N} = \{m : m = N(k \cap \text{Bd } E_i) \text{ for infinitely many } i\}.$$

Then $P_0(k, p)$ is the smallest element of \mathcal{N} ; if \mathcal{N} is empty, $P_0(k, p)$ is infinite.

Proof. This is proved by using the corollary to Theorem 1; one only has to show that $P_0(k, V_s - V_{s+1}) = N(k \cap \text{Bd } E_{s+1})$ for each index s .

Let C be a tame closed 3-cell neighbourhood of p such that

(i) $\text{Bd } C \subset \text{Int}(V_s - V_{s+1})$ and is in general position with respect to $\text{Bd } E_{s+1}$, meeting $\text{Bd } E_{s+1}$ in a finite number of simple closed curves which do not contain any points of k , and

(ii) $N(k \cap \text{Bd } C) = P_0(k, V_s - V_{s+1})$.

We may choose an intersection curve σ which bounds a disc $D \subset \text{Bd } E_{s+1}$ which contains no other intersection curves. σ separates $\text{Bd } C$ into two discs; one of these, say D' , in union with D forms the boundary of a 3-cell S which does not contain V_{s+1} . S lies in $\text{Int}(V_s - V_{s+1})$.

$D \cup (\text{Bd } C - D')$ is the boundary of a 3-cell neighbourhood of p , and lies in $\text{Int}(V_s - V_{s+1})$; $D \cup (\text{Bd } C - D')$ must therefore meet k in at least $P_0(k, V_s - V_{s+1})$ points, and it follows that $N(k \cap D) \geq N(k \cap D')$ because $N(k \cap \text{Bd } C)$ is minimal.

Suppose $N(k \cap D) > N(k \cap D')$; then there exists a subarc α of k with $\text{Int } \alpha \subset \text{Int } S$, $\alpha \cap \text{Bd } E_{s+1} \subset \text{Int } D$, and whose endpoints lie in $\text{Int } D$. We join the endpoints of α by an oriented arc $\alpha' \subset \text{Int } D$, so that $\alpha \cup \alpha'$ is a simple closed curve in S .

Either $\alpha \in A(E_{s+1}, E_{s+2})$ or $\alpha \in B(E_{s+1}, E_s)$. If $\alpha \in A(E_{s+1}, E_{s+2})$, then each pair (α, β) is splittable, where $\beta \in B(E_{s+2}, E_{s+1})$; for

$$\beta \cup E_{s+2} \subset \text{Int } V_{s+1} \subset V_{s+1} \subset V_s - N,$$

where N is a suitably small tame closed regular neighbourhood of S . Therefore the arc α cannot lie in $A(E_{s+1}, E_{s+2})$, for this would contradict the fact that $E_{s+1} \supset V_{s+1} \supset E_{s+2}$ is part of a special constructing sequence for the exceptional arc k .

α cannot lie in $B(E_{s+1}, E_s)$ either, for $\alpha \cup \alpha' \subset \text{Int } N \subset \text{Int } V_s \subset V_s \subset E_0 - \beta \cup \text{Bd } E_s$ for each $\beta \in A(E_s, E_{s+1})$, thus implying that each pair (β, α) is splittable and contradicting the fact that $E_s \supset V_s \supset E_{s+1}$ is part of a special constructing sequence for k .

Therefore, no subarc α of k with $\alpha \cap \text{Bd } E_{s+1} \subset \text{Int } D$ and whose endpoints lie in $\text{Int } D$ can have its interior lying entirely in $\text{Int } S$; that is, α meets D' whenever α meets D , so $N(k \cap D') \geq N(k \cap D)$.

It follows that $N(k \cap D) = N(k \cap D')$, and $D \cup (\text{Bd } C - D')$ is the boundary of a 3-cell neighbourhood C' of p , lies in $\text{Int}(V_s - V_{s+1})$, and meets k in $P_0(k, V_s - V_{s+1})$ points. We put C' into general position with respect to $\text{Bd } E_{s+1}$ by taking $C'' = C' - \{\text{an open regular neighbourhood of } D\}$ if $D \subset C$, or by taking $C'' = C' \cup \{\text{a closed regular neighbourhood of } D\}$ if $D \subset \text{Cl}(E_0 - C)$. By choosing sufficiently small regular neighbourhoods, we may ensure that

- (i) $\text{Bd } C''$ meets k in $P_0(k, V_s - V_{s+1})$ points, and lies in $\text{Int}(V_s - V_{s+1})$;
- (ii) $\text{Bd } C''$ is in general position with respect to $\text{Bd } E_{s+1}$, and $\text{Bd } C'' \cap \text{Bd } E_{s+1} \subset \text{Bd } C \cap \text{Bd } E_{s+1} - \{\sigma\}$.

In particular, we have eliminated all those curves of $\text{Bd } C \cap \text{Bd } E_{s+1}$ which lie in D' .

After finitely many such "cutting and pasting" operations, therefore, we may replace C by a 3-cell neighbourhood C^* of p , whose boundary lies in $\text{Int}(V_s - V_{s+1})$, meets k in $P_0(k, V_s - V_{s+1})$ points, and does not meet $\text{Bd } E_{s+1}$.

Now either $C^* \subset \text{Int } E_{s+1}$, or $E_{s+1} \subset \text{Int } C^*$. We will show that the first of these is impossible if $P_0(k, V_s - V_{s+1}) < N(k \cap \text{Bd } E_{s+1})$; a similar proof will show that the second is impossible if $P_0(k, V_s - V_{s+1}) < N(k \cap \text{Bd } E_{s+1})$.

If $P_0(k, V_s - V_{s+1})$ is smaller than $N(k \cap \text{Bd } E_{s+1})$, and $C^* \subset \text{Int } E_{s+1}$, there must exist an arc $\alpha \in A(E_{s+1}, E_{s+2})$ which lies in $E_{s+1} - C^*$. But then the pair (α, β) is splittable for every $\beta \in B(E_{s+2}, E_{s+1})$, because

$$\beta \cup \text{Bd } E_{s+2} \subset \text{Int } C^* \subset C^* \subset E_0 - \alpha \cup \text{Bd } E_{s+1}.$$

Since k is exceptional, this is impossible and no such arc α can exist; that is,

$$P_0(k, V_s - V_{s+1}) = N(k \cap \text{Bd } E_{s+1}).$$

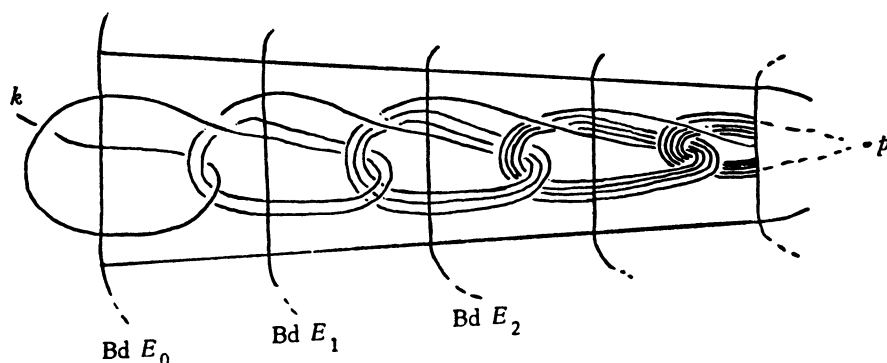
Similarly, $P_0(k, V_s - V_{s+1}) = N(k \cap \text{Bd } E_{s+1})$ if $E_{s+1} \subset \text{Int } C^*$.

This proves the theorem. Q.E.D.

Theorem 2 yields the following corollaries, which are very similar to Theorems 5 and 6 of [8].

Theorem 3. *If $m \geq 1$ is an integer, there exists a nearly polyhedral arc k such that $\text{LEG}(k, p) = 1$ and $P_0(k, p) = 2m + 1$.*

Theorem 4. *There exists a nearly polyhedral arc k such that $\text{LEG}(k, p) = 1$ and $R_0(k, p) = \aleph_0$.*



$$N(k \cap \text{Bd } E_s) = 2s + 3, s = 0, 1, 2, \dots$$

Figure 2

The arc shown in Figure 2 has $\text{LEG}(k, p) = 1$ and $P_0(k, p) = \aleph_0$.

Added in proof. In a preprint titled *The penetration index of a self-linked nearly tame arc*, Carl D. Sikkema has generalised the results of this paper to the wider class of self-linked nearly tame arcs. Sikkema's methods are quite independent of those used here, and he does not make any restriction on the total penetration index of his arcs.

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